Optimizing a Wooden Pile-and-Plank Retaining Wall

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OBJECTIVE

This project was developed to optimize the cost for a wooden pile-and-plank retaining wall with a total length of 80' and a height of 5'. The wall will be constructed out of Standard Structural Timber. A pressure of 500 [$\frac{lb_f}{ft^2}$] acts at the base of the wall and decreases linearly to 100 [$\frac{lb_f}{ft^2}$]. The piles must extend 5 [ft] below and above grade. Therefore the total height of the piles is 10 [ft]. The allowable flexural stress of the timber is 1,200 [psi] which is equivalent to 172,800 [psf]. It is important to note, standard structural timbers are only available in 8', 10', and 12' lengths. Only the piles with a cross section of 4"x4", 6"x6", 8"x8", 10"x10", and 12"x12" were considered. The timber costs $14 per cubic feet and each pile needs concrete footing which costs $40.

NOMENCLATURE

Symbols	Definition	Units
σ	Flexural Stress	[Pa]
C	Cost	[$]
CL	Concentrated Load	[lb_f]
h	Height	[ft]
L	Length	[ft]
Len	Length of Plank Purchased	[ft]
M	Moment	[$ft - lb_f$]
n	Number of Piles	[-]
Q	Number of Planks Needed	[-]
R	Reaction Force	[lb_f]
S	Section Modulus	[ft^3]
w	Width	[ft]
y_c	Location of Centroid	[ft]

TABLE I
SYMBOLIC DEFINITIONS

METHODOLOGY

To optimize the cost; equations for the piles, and planks were derived separately and optimized within a Matlab code.

PILES

When analyzing the piles, the objective was to find an equation for the minimum section modulus in terms of the length (L). The section modulus was used to find the cross sectional area of the piles. This will be discussed later.

To begin, the wall needs to be a total length of 80 [ft], therefore the equation for the length of a plank is listed below.

$$L = \frac{80[ft]}{n-1} \tag{1}$$

L is the length while n is the number of piles.

Moving on, it can be assumed that the piles are cantilever beams. When drawing the free body diagram (FBD), the concentrated load (CL) was determined by finding the area of a trapezoid with equation 2.

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$$CL = \frac{100 * L[\frac{lb_f}{ft^2}] + 500 * L[\frac{lb_f}{ft^2}]}{2} * 5[ft] \tag{2}$$

$CL = 1500[\frac{lb_f}{ft}] * L$

The concentrated load will act on the centroid of the trapezoid. While following a YouTube video published by Civil Engineering the equation for the centroid is determined below.

$$y_c = \frac{500 * L[\frac{lb_f}{ft^2}] + 2 * L * 100[\frac{lb_f}{ft^2}]}{3(100 * L[\frac{lb_f}{ft^2}] + 500 * L[\frac{lb_f}{ft^2}])} * 5[ft] \tag{3}$$

$y_c = \frac{35}{18}[ft]$

With the CL and the centroid determined, the FBD is shown below.

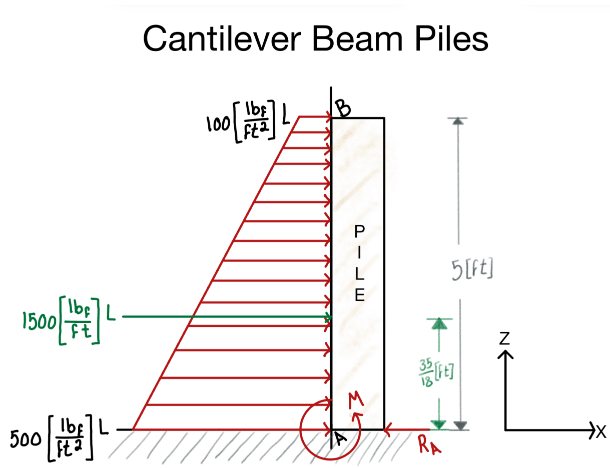

Fig. 1. FBD of a Pile as a Cantilever Beam

In reference to figure 1, the equations for the summation of forces in the x direction and the moment about point A are listed below.

$$\sum F_x = R_A - 1,500[\frac{lb_f}{ft}] * L = 0[lb_f] \tag{4}$$

$$\sum M_A = M - \frac{35}{18}[ft] * 1,500[\frac{lb_f}{ft}] * L = 0[ft - lb_f] \tag{5}$$

$R_A = 1,500[\frac{lb_f}{ft}] * L$
$M = \frac{8,750}{3}[lb_f] * L$

R_A being the reactionary force at A. M is the reactionary moment at A.

To derive the equation for the minimum section modulus in terms of L, it was important to only consider the absolute value of σ_{max}, M_{max}, and S_{min}.

$$\sigma_{max} = \frac{M_{max}}{S_{min}} \tag{6}$$

The σ_{max} is the maximum flexural stress which was given as 172,800 [psf]. M_{max} is the absolute value of M. Lastly, S_{min} is the minimum section modulus. After rewriting equation 6 and substituting the known values, the equation for the minimum section modulus is listed below.

$$S_{min} = \frac{(\frac{8,750}{3}[lb_f] * L)}{172,800[psf]} \tag{7}$$

Like previously mentioned, the S_{min} value will determine the cross sectional area of the piles by comparison. The table below lists the section moduli values which were drawn from a handout given by the instructor.

Cross-Sectional Area	Section Modulus
4"X4"	0.0045949 $[ft^3]$
6"X6"	0.01603 $[ft^3]$
8"X8"	0.040683 $[ft^3]$
10"X10"	0.082755 $[ft^3]$
12"X12"	0.14641 $[ft^3]$

PLANKS

In order to determine the possible sizes for the planks, the minimum width (w) needed to be found in terms of L. This was ultimately done using equation 6.

Starting with finding the maximum bending moment, a free body diagram was made. In this case, the planks are assumed to be simply supported with a pin at one end of the plank and a roller at the other. Due to the fact that the pressure given was for a three dimensional plank, $500\frac{lb_f}{ft^2}$, and it was being analyzed in a two dimensional orientation, the pressure was multiplied by the height of the plank (h), so that there was a distributed load across the plank in $\frac{lb_f}{ft}$. Additional analysis shows that there were also two reactionary forces, one at either end, as shown below.

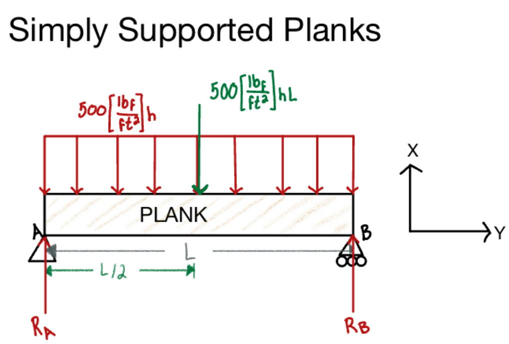

Fig. 2. FBD of a Simply Supported Plank

After constructing a FBD, the moments and the forces in the y direction were then balanced so that the equations for the reactionary forces can be found.

$$\sum F_y = R_A + R_B - (500[\frac{lb_f}{ft^2}] * h * L) = 0 \tag{8}$$

$$\sum M_A = (L * R_B) - \frac{(500[\frac{lb_f}{ft^2}] * h * L^2)}{2} = 0 \tag{9}$$

$R_A = 250[\frac{lb_f}{ft^2}] * h * L$
$R_B = 500[\frac{lb_f}{ft^2}] * h * \frac{L}{2}$

Once the equations for the reactionary forces were found, diagrams of the distributed load, shear force, and bending moment were made in order to find the maximum bending moment.

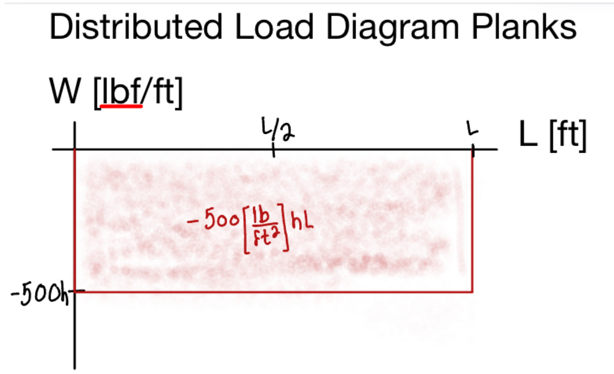

Fig. 3. Distributed Load Diagram of Planks

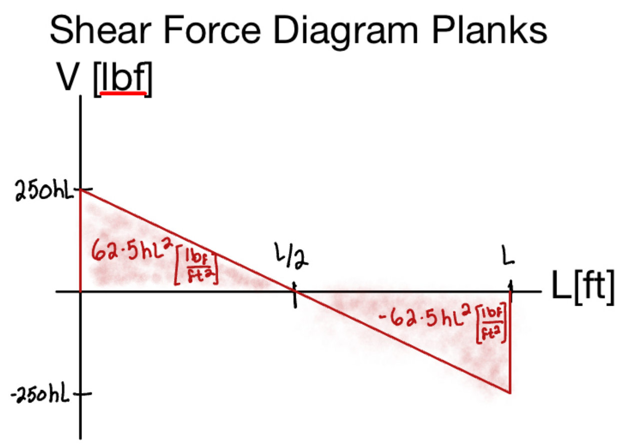

Fig. 4. Shear Force Diagram of Planks

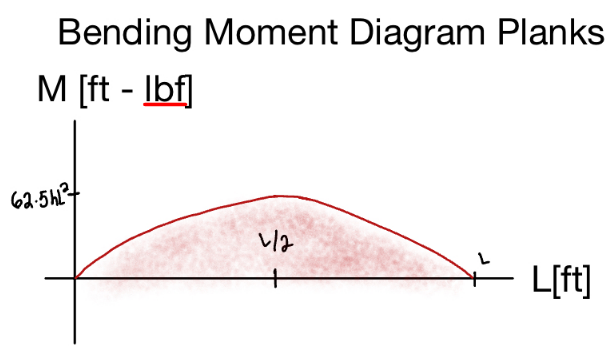

Fig. 5. Bending Moment Diagram of Planks

Through this method, it was found that the maximum bending moment is

$$M_{max} = \frac{1}{4} * 250[\frac{lb_f}{ft^2}] * h * L^2 \tag{10}$$

With the maximum bending moment found, the next thing that needed to be calculated is the minimum section modulus. The section modulus can be found using the equation

$$S = I/y_c \tag{11}$$

The second moment of the plank can then be given as

$$I = \frac{1}{12} * h * w^3 \tag{12}$$

and the centroid can be given as

$$y_c = \frac{1}{2} * w \tag{13}$$

Plugging equations 12 and 13 into the equation 11, the minimum section modulus equation results in the equation listed below.

$$S_{min} = \frac{1}{6} * h * w^2 \tag{14}$$

Finally, everything was plugged into the equation for stress, with maximum stress equal to 172,800 [$\frac{lb}{ft^2}$], the equation was found as

$$172,800[\frac{lb}{ft^2}] = \frac{\frac{1}{4} * 250[\frac{lb_f}{ft^2}] * h * L^2}{\frac{1}{6} * h * w^2} \tag{15}$$

Which, when solving for minimum width, was simplified to

$$w_{min} = \sqrt{\frac{750 * L^2}{345600}} \tag{16}$$

This minimum width equation could then be compared to the width of the planks that could be bought and used to find the smallest plank that could be used for any given length of a plank.

CREATING A MATRIX OF POSSIBLE LENGTHS

A Matlab script was written to iterate through the number of piles starting at 7, which was the lowest feasible number considering the largest length for a plank can be 12'. Therefore, with equation 7 a plot of possible plank lengths v. minimum section moduli was created (figure 6). With equation 16, a plot of possible plank lengths v. plank width was created (figure 7).

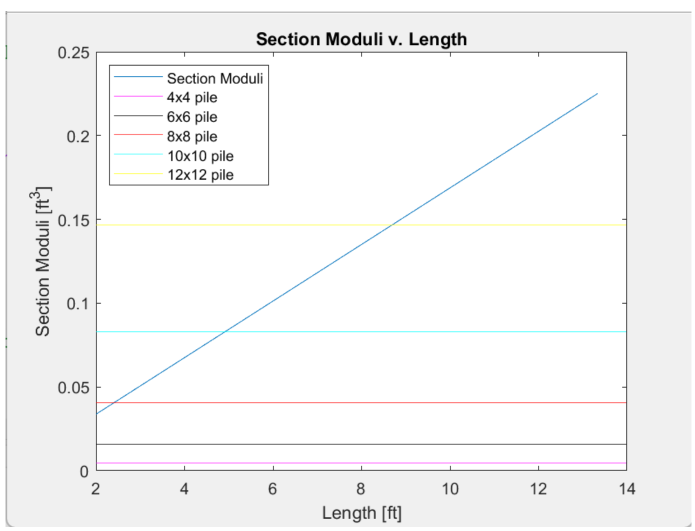

Fig. 6. Section Moduli v. Length

Figure 6 shows the section moduli for the standard piles which fall on the part of the graph that is greater than or equal to the minimum section modulus. Therefore implying those sizes are usable. In this case, the 4"x4" and 6"x6" piles fall solely on the part of the graph that is less than the minimum section modulus and cannot be used no matter what length of plank is used. The 4"x4" and 6"x6" piles cannot withstand the pressure exerted by the soil. Thus, only 8"x8", 10"x10", and 12"x12" piles were considered moving forward.

From figure 7 shows the width for the standard structural timber lengths which fall on the part of the graph that is greater than or equal to the width. Therefore implying those sizes are usable. It can be determined that a width greater than 6" is

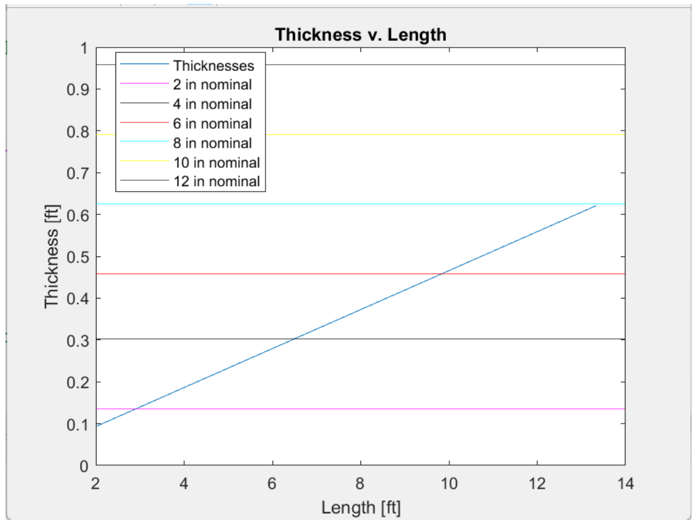

Fig. 7. Plank width v. Length

usable, however it is unnecessary for the parameters given within the project. Thus, only the analysis of 2”, 4” and 6” planks were considered moving forward.

It is important to note that the height of the planks were determined to be independent of the optimized length. Thus the height was agreed to be 7.5” due to the fact it was easily divisible by the total height above grade, 60”.

Using equations 1, 5, 7 and 16 in Matlab, a matrix of possible solutions was created. Any solutions exceeding the constraints set by the section modulus for the 12”x12” post and the width for a 12’ long plank were not considered. The calculated minimum width in the matrix were not of tangible beams however. Therefore the width for each possible configuration were rounded up to the nearest width of a tangible beam. For example, if a minimum width was calculated to be 2”, it would be rounded up to the next highest tangible beam width: 3.625” which is the width of a _” x 4” beam. Similarly, the calculated minimum section moduli in the matrix did not match the section moduli for the given cross sectional areas of the piles. Therefore the section moduli for each possible configuration were rounded up to the nearest section modulus.

CALCULATING COST

Once all possible lengths and the associated width, moments, and cross-sectional areas were found, these values were inputted into a cost function. In order to ensure the least amount of wood was wasted, the cost function took the possible lengths of the planks and found the remainder when divided by 8’, 10’ and 12’. Whichever length yielded the smallest remainder was the length to be used in building the wall. The number of planks needed to build to a total length of 80 was calculated by:

$$Q = \frac{80[ft]}{Len - R} \tag{17}$$

Where Q is the number of planks needed to be purchased, Len is the length of the plank purchased and R is the smallest calculated remainder calculated. The cost of plank wood [C1] was then calculated using the formula below:

$$C1 = Len * Q * 5[ft] * w * 14[\frac{\$}{ft^3}] \tag{18}$$

Where w is the minimum width from the matrix, 5 is the height in feet of the wall above grade and 14 is the cost per volume of wood purchased.

The cost of the piles, C2, including the concrete footing was then calculated using the equation below:

$$C2 = (Area * 10[ft] * 14[\frac{\$}{ft^3}]) + (n * 40[\$]) \tag{19}$$

Where *Area* is the cross-sectional area of the pile from the matrix, n is the number of piles, 10 is the total height of the piles, and 40 is the cost of the concrete needed for each pile.

The total cost, C, was found to be the sum of the cost of the piles and the cost of the planks.

$$C = C1 + C2 \tag{20}$$

A plot was then created comparing the distribution of the total cost for a given number of piles. This is shown below.

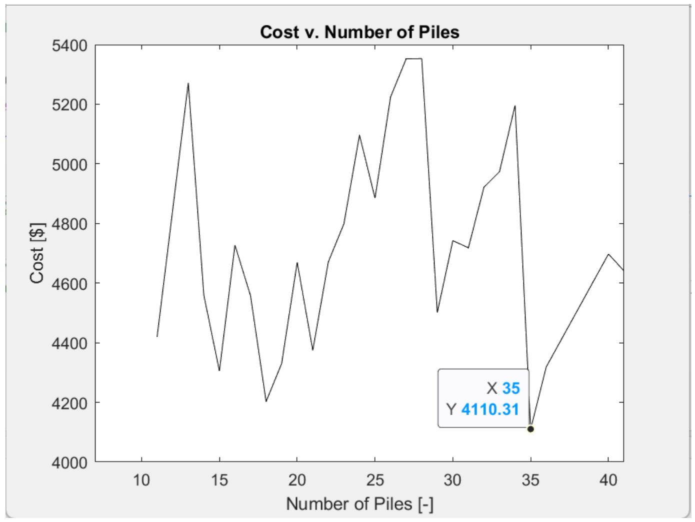

Fig. 8. Cost v. Number of Piles

RESULTS AND DISCUSSION

COST OF PLANKS

For this part, the focus will be on the optimized case, which is highlighted in the figure above. From visual inspection, it is seen that the cost is optimized when the number of piles is set equal to 35. All other cases are easily calculated using the methods outlined below but are omitted due to redundancy.

To start, the length was calculated using equation 1, which has a value of 2.353 [ft]. After finding the length, the minimum width was solved for using equation 16. It has a calculated value of 0.13542 [ft]. From there, the 8', 10', and 12', remainders were calculated as 0.941, 0.588, 0.235, respectively. Since the remainder of the 12' was the least, Len was set to 12'. Then, using equation 17, the number of planks needed was calculated, which resulted in a value of 7, only across the length of the wall once. Since the height of the planks is 7.5", Q was repeated 8 times to reach a total height of 60". All the variables needed for equation 18 were found and C1 was calculated as $796.27.

COST OF PILES AND CONCRETE

Once the final cost of the planks was calculated, the calculations then shifted to the cost of the piles and concrete. Equation 19 will be utilized to calculate the cost of piles and concrete. The only unknown at this point was the cross-sectional area. Like previously mentioned, to find this area, the section moduli for each possible configuration were rounded up to the nearest section modulus of the given cross sectional areas of the piles. It was determined that the best area used for this calculation would be an 8"x8" pile, which has a dressed size of 7.5"x7.5". This resulted in an area of 56.25 $[in^2]$, or 0.391 $[ft^2]$. Since all of the variables were then known, equation 19 was used, and C2 was found to have a final value of $3,314.04.

TOTAL COST

Using all of the calculated values above, and equation 20, it was found that the final calculated cost of the wall is $4,110.31.

BILL OF MATERIALS

Following the cost calculations, a Bill of Materials was put together highlighting the items used, unit price, quantity of the materials, and the total price of the selected material. This is all seen in the figure below.

BILL OF MATERIALS				
Item #	Description	Unit Price	Quantity	Total Price
1	Piles (8X8X10)	$14.00	35	$1,914.04
2	Planks (2 X 8 X 12)	$14.00	56	$796.27
3	Concrete Footing	$40.00	35	$1,400.00
			Total Cost:	$4,110.31

Fig. 9. Bill of Materials

FINAL RESULTS

To reiterate, the most cost effective design includes 35 piles, with a dressed cross section of 7.5”X7.5”. The length of the planks needs to be 2.353 [ft], with a width of $1\frac{5}{8}$”. With these dimensions, the final cost is found to be $4,110.31.

ACKNOWLEDGEMENTS

For the duration of this project, the team received insightful feedback from our professor, Dr. William S. Slaughter IV. We would like to thank him for all the time he spent working with us to aid us in improving our design. We would like to thank all the teaching assistants for their continued support.

REFERENCES

[1] Civil Engineering (2019) How to find Centroid of Trapezoid — Example Solved. [online] Available at: https://www.youtube.com/watch?v=-sJDag1H0t4&t=3s